Double-angle identities

Very often we'll handle trig functions with an angle of 2θ, instead of just θ, like $\sin 2\theta$. We'll usually want to convert these to get the 2 out of the angle. The way to do it is using double-angle identities. We can't simply say $\sin 2\theta = 2\sin\theta$; that's just not true. 2θ is the argument of the sine function; we're evaluating the sine function at the angle 2θ, so the $\sin$ and 2θ aren't simply multiplied together such that we can pull the 2 out in front.

The double-angle identities are actually derived from the sum identities we learned about earlier. For example, if we start with the sum identity for cosine

$$\cos(\theta + \alpha) = \cos\theta\cos\alpha - \sin\theta\sin\alpha$$

and letting $\alpha = \theta$, we get

$$\cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

This is the double-angle identity for the cosine function. We can similarly find the other double angle identities. Here are all three double angle identities:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

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Example

If θ is an angle in the second quadrant with $\cos\theta = -\sqrt{5}/6$, find $\cos 2\theta$ and $\sin 2\theta$.

By the double-angle identity for cosine,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

In order to apply this identity, we first have to find $\sin^2\theta$. By the basic Pythagorean identity,

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta$$

Since $\cos\theta = -\sqrt{5}/6$, we get

$$\sin^2\theta = 1 - \left(\frac{\sqrt{5}}{6}\right)^2$$

$$\sin^2\theta = 1 - \left(\frac{5}{36}\right) = \frac{31}{36}$$

Plus, since we know that $\cos\theta = -\sqrt{5}/6$, we can say

$$\cos^2\theta = \left(-\frac{\sqrt{5}}{6}\right)^2 = \frac{5}{36}$$

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Substituting $\cos^2\theta = 5/36$ and $\sin^2\theta = 31/36$ into the double-angle identity for cosine, we get

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = \frac{5}{36} - \frac{31}{36}$$

$$\cos 2\theta = -\frac{26}{36} = -\frac{13}{18}$$

To find $\sin 2\theta$ for this same angle θ, we'll use the double-angle identity for sine.

$$\sin 2\theta = 2\sin\theta\cos\theta$$

We've already found that

$$\sin^2\theta = \frac{31}{36}$$

but we don't yet know the value of $\sin\theta$. Since θ is in the second quadrant, $\sin\theta$ is positive. Therefore,

$$\sin\theta = \sqrt{\frac{31}{36}} = \frac{\sqrt{31}}{\sqrt{36}} = \frac{\sqrt{31}}{6}$$

Now we're ready to apply the double-angle identity for sine.

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin 2\theta = 2\left(\frac{\sqrt{31}}{6}\right)\left(-\frac{\sqrt{5}}{6}\right)$$

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$$\sin 2\theta = -\frac{2\sqrt{31}\sqrt{5}}{36} = -\frac{\sqrt{155}}{18}$$

Next we'll take a look at how to apply the double-angle identities when we know the value of $\sin\theta$.

Example

Find $\sin 2\theta$ and $\cos 2\theta$ for an angle θ which lies in the third quadrant and has $\sin\theta = -3/7$.

By the double-angle identity for sine,

$$\sin 2\theta = 2\sin\theta\cos\theta$$

We already know that $\sin\theta = -3/7$, but we need to find $\cos\theta$. By the basic Pythagorean identity,

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

Substituting $-3/7$ for $\sin\theta$ gives

$$\cos^2\theta = 1 - \left(-\frac{3}{7}\right)^2$$

$$\cos^2\theta = 1 - \left(\frac{9}{49}\right) = \frac{40}{49}$$

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$$\cos\theta = \pm\sqrt{\frac{40}{49}}$$

$$\cos\theta = \pm\frac{\sqrt{40}}{7}$$

Since θ is in the third quadrant, we know that $\cos\theta$ is negative, so we can ignore the positive value and say

$$\cos\theta = -\frac{\sqrt{40}}{7}$$

Substituting $\cos\theta = -\sqrt{40}/7$ and $\sin\theta = -3/7$ into the double-angle identity for sine, we get

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin 2\theta = 2\left(-\frac{3}{7}\right)\left(-\frac{\sqrt{40}}{7}\right)$$

$$\sin 2\theta = \frac{6\sqrt{40}}{49}$$

By the double-angle identity for cosine, we get

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = \left(-\frac{\sqrt{40}}{7}\right)^2 - \left(-\frac{3}{7}\right)^2$$

$$\cos 2\theta = \frac{40}{49} - \frac{9}{49} = \frac{31}{49}$$

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If we're given just the value of $\tan\theta$ for some angle θ, we can compute the value of $\tan 2\theta$ (even if we don't know the quadrant in which θ lies) by using the double-angle identity for tangent.

Example

Find $\tan 2\theta$ if $\tan\theta = \sqrt{23}$.

By the double-angle identity for tangent,

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\tan 2\theta = \frac{2\sqrt{23}}{1-\left(\sqrt{23}\right)^2}$$

$$\tan 2\theta = \frac{2\sqrt{23}}{1-23} = \frac{2\sqrt{23}}{-22} = -\frac{\sqrt{23}}{11}$$

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